A QUINTIC HYPERSURFACE IN P^4 WITH 130 NODES

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INTRODUCTION

The maximum number $N_n(d)$ of ordinary double points (nodes, for short) a hypersurface of degree d in P^n can have, has been the subject of investigation of several authors. (We refer to [1] for an overview.) The best upperbound so far known for $n \geq 4$ is the so-called *spectral bound*, obtained by Varchenko ([13]). For hypersurfaces in P^4 of low degree one obtains from this:

$$N_4(3) \leq 10$$
$$N_4(4) \leq 45$$
$$N_4(5) \leq 135$$

In fact, $N_4(3) = 10$, realised by the Segre Cubic ([11]), and also $N_4(4) = 45$, realized by the Burkhardt Quartic ([4]). These remarkable threefolds are uniquely determined by having these singularities (see [12], [9], and [8]). There is a very rich geometry associated with these varieties (see e.g. [12], [2], [5]).

In [7] Hirzebruch constructed a quintic with 126 nodes. In this note we present a quintic $\mathcal{M}$ in P^4 with 130 nodes, thus narrowing down the possibilities for $N_4(5)$ to:

$$130 \leq N_4(5) \leq 135$$

§1. THE QUINTIC

Consider the space P^5, with homogeneous coordinates $X_0, X_1, \ldots, X_5$. Let

$$S_i := S_i(X_0, X_1, \ldots, X_5)$$

be the *i*-th elementary symmetric function in the X_i. The equation

$$S_1 = X_0 + X_1 + X_2 + X_3 + X_4 + X_5 = 0$$

defines a $P^4 \subset P^5$, on which the symmetric group Σ_6 acts by permutation of the coordinates.

THEOREM 1. *The quintic hypersurface $\mathcal{M} \subset P^4$ defined by the equations*

$$S_1 = 0$$
$$S_5 + S_2 S_3 = 0$$

has exactly 130 *nodes. These form three orbits under the permutation group* Σ_6. *The following table gives a name, the number of elements and a representative point of each* Σ_6-*orbit.*

Name	Number of elements	Point of Σ_6-orbit
Segre	10	$(1:1:1:-1:-1:-1)$
Moving	90	$(1:1:-1:-1:\sqrt{-3}:-\sqrt{-3})$
Extra	30	$(1:1:1:1:\sqrt{-3}-2:-\sqrt{-3}-2)$

§2. PROOF OF THEOREM 1

The space of Σ_6-invariant quintics in the P^4 defined by $S_1 = 0$ is spanned by S_5 and $S_2 S_3$. For each quintic in the pencil we shall determine the singular locus. For this purpose it is convenient to express the S_i in terms of the *power sums*:

$$C_k := C_k(X_0, X_1, \ldots, X_5) = \sum_{i=0}^{5} X_i^k.$$

Modulo S_1 one has:

$$S_2 = -\tfrac{1}{2}C_2$$

$$S_3 = \tfrac{1}{3}C_3$$

$$S_4 = -\tfrac{1}{4}C_4 + \tfrac{1}{8}C_2^2$$

$$S_5 = \tfrac{1}{5}C_5 - \tfrac{1}{6}C_2 C_3.$$

Consider the variety $\mathcal{M}_{(\alpha:\beta)} \subset P^4$ defined by:

$$F_{(\alpha:\beta)} := \alpha S_5 + \beta S_2 S_3 = \frac{\alpha}{5} C_5 - \frac{(\alpha + \beta)}{6} C_2 C_3.$$

So, $\mathcal{M} = \mathcal{M}_{(1:1)}$ is defined by $S_5 + S_2 S_3 = \tfrac{1}{5}C_5 - \tfrac{1}{3}C_2 C_3 = 0$.

A point $(\eta) := (\eta_0 : \eta_1 : \ldots : \eta_5) \in P^4 \subset P^5$ of $\mathcal{M}_{(\alpha:\beta)}$ is a *singular point* if and only if the differentials of the two defining equations in P^5 are dependent, i.e. for some $(v:\mu) \in P^1$ one has:

$$v \partial_i F_{(\alpha,\beta)}(\eta) = \mu \partial_i S_1(\eta).$$

Clearly, $v \neq 0$, so we can put $v = 1$. Furthermore, the variety $\mathcal{M}_{(0:1)}$ has clearly the surface $S_2 = S_3 = 0$ as singular locus. So we assume that $\alpha \neq 0$ and put $\lambda := (\alpha + \beta)/2\alpha$. From the definition of $F_{(\alpha:\beta)}$, and by summing over all indices to eliminate μ, it follows that each coordinate η_i has to satisfy the equation:

$$P_\lambda := X^4 - \lambda C_2 X^2 - \frac{2}{3} \lambda C_3 X - \frac{1}{6}(C_4 - \lambda C_2^2)$$

$$= X^4 - \frac{1}{6} C_4 - \lambda \left(C_2 X^2 - \frac{1}{6} C_2^2 + \frac{2}{3} C_3 X \right)$$

$$= 0$$

where now $C_i = C_i(\eta_0, \eta_1, \ldots, \eta_5)$. Now let x, y, z, t be the four roots of the polynomial $P_\lambda(X)$, where we consider λ, C_2, C_3, C_4 as *variable constants*. Note that from the form of P_λ

it follows that $x + y + z + t = 0$. *A priori* there are nine different ways in which the coordinates η_i of a singular point of $\mathcal{M}_{(\alpha:\beta)}$ can be distributed over these four roots. Below we list these cases

Case 1:	$6x$	Case 4:	$3x, 3y$	Case 7:	$2x, 2y, 2z$
Case 2:	$5x, y$	Case 5:	$4x, y, z$	Case 8:	$3x, y, z, t$
Case 3:	$4x, 2y$	Case 6:	$3x, 2y, z$	Case 9:	$2x, 2y, z, t$

We analyse case by case:

Case 1. This can not occur, since $(\eta) = (x:x:\ldots:x) \in P^4$, so x would have to be zero.

Case 2. We may assume $(\eta) = (1:1:1:1:1:-5)$. Hence $C_2 = 30, C_3 = -120, C_4 = 630$ and

$$P_\lambda = (X^4 - 105) - \lambda(30X^2 - 80X - 150).$$

If we require $P_\lambda(1) = P_\lambda(-5) = 0$, we find $\lambda = \frac{13}{25}$ and thus $(\alpha:\beta) = (25:1)$.

Case 3. We may assume $(\eta) = (1:1:1:1:-2:-2)$. Hence $C_2 = 12, C_3 = -12, C_4 = 36$ and

$$P_\lambda = (X^4 - 6) - \lambda(12(X^2 - 2) - 8X).$$

If we require $P_\lambda(1) = P_\lambda(-2) = 0$, we find $\lambda = \frac{1}{4}$ and $(\alpha:\beta) = (2:-1)$.

Case 4. We may assume $(\eta) = (1:1:1:-1:-1:-1)$. Hence $C_2 = 6, C_3 = 0, C_4 = 6$ and

$$P_\lambda = (X^4 - 1) - 6\lambda(X^2 - 1).$$

For *all* λ we have $P_\lambda(1) = P_\lambda(-1) = 0$.

Case 5. We may assume that (η) has the form

$$(\eta) = (x:x:x:x:u-2x:-u-2x)$$

For $x = 0$ we find immediately that $P_\lambda(0) = P_\lambda(1) = P_\lambda(-1) = 0$ occurs only for $\lambda = \frac{1}{2}$. So we may assume that $(\eta) = (1:1:1:1:u-2:-u-2)$. Hence $C_2 = 12 + 2u^2$, $C_3 = -12 - 12u^2, C_4 = 36 + 48u^2 + 2u^4$. Equating $P_\lambda(X)$ to zero for $X = 1, X = u - 2$, $X = -u - 2$ leads to the equations:

$$u^4 + 24u^2 + 15 = 2\lambda(u^4 + 21u^2 + 30)$$

$$u^4 - 12u^3 + 24u^2 - 48u + 15 = 2\lambda(u^4 - 12u^3 + 21u^2 - 42u + 30)$$

$$u^4 + 12u^3 + 24u^2 + 48u + 15 = 2\lambda(u^4 + 12u^3 + 21u^2 + 42u + 30).$$

The sum of the last two gives the first equation, whereas substraction leads to:

$$u(u^2 + 4) = u\lambda(2u^2 + 7).$$

The solution $u = 0$ brings us back to Case 3, so we assume $u \neq 0$. Hence $0 = (2\lambda - 1)u^2 + (7\lambda - 4)$, which in combination with the first equation gives as solutions:

$$\lambda = 1, \quad u = \pm\sqrt{-3}$$

$$\lambda = \tfrac{13}{25}, \quad u = \pm 3 \qquad \text{(Case 2)}.$$

Case 6. We may assume that $(\eta) = (x:x:x:u - x:u - x: -2u - x)$. The case $x = 0$ leads to $\lambda = \frac{1}{2}$, i.e. $(\alpha:\beta) = (1:0)$, so we take $x = 1$. One has: $C_2 = 6(u^2 + 1)$, $C_3 = -6u^2(u + 3)$, $C_4 = 6(3u^4 + 4u^3 + 6u^2 + 1)$. Equating $P_\lambda(X)$ to zero for $X = 1, X = u - 1$ leads to:

$$u^2(3u^2 + 4u + 6) = 2u^2\lambda(3u^2 + 2u + 9)$$

$$u(u^3 + 4u^2 + 2) = 2u\lambda(u^3 + 5u^2 - 3u + 3).$$

The determinant of this equation system for λ is

$$20u^4(u + 1)^2(u - 2).$$

The solutions $u = 0$, $u = 2$, $u = -1$ lead us back into the Cases 4, 2 and 3, respectively.

Case 7. We may assume that $(\eta) = (x:x:y:y:z:z)$ with $x + y + z = 0$. This implies that the fourth root t of P_λ has to be zero. If we denote by ε_i the i-th power sum in and by σ_i the i-th elementary symmetric function in x, y, z, we can write $C_i = 2\varepsilon_i$ and so:

$$P_\lambda = X(X^3 - 2\varepsilon_2\lambda X - \tfrac{4}{3}\varepsilon_3\lambda) - \tfrac{1}{3}(\varepsilon_4 - 2\lambda\varepsilon_2^2).$$

So we obtain the following equations:

$$
\begin{aligned}
\varepsilon_4 - 2\lambda\varepsilon_2^2 &= 0 \\
2\lambda\varepsilon_2 &= \sigma_2 = -\tfrac{1}{2}\varepsilon_2 \\
-\tfrac{4}{3}\lambda\varepsilon_3 &= -\sigma_3 = -\tfrac{1}{3}\varepsilon_3 \\
0 &= \varepsilon_4 - 2\lambda\varepsilon_2^2 = (\tfrac{1}{2} - 2\lambda)\varepsilon_2^2.
\end{aligned}
$$

We conclude that if $\lambda = \frac{1}{4}$, then there are no additional conditions on x, y, z, whereas $\lambda \neq \frac{1}{4}$ leads to $\varepsilon_2 = \varepsilon_3 = 0$, which implies $x = y = z = 0$.

Case 8. We may assume $(\eta) = (x:x:x:y:z:t)$, with $3x + y + z + t = 0$ but because $x + y + z + t = 0$, we obtain $x = 0$, $y + z + t = 0$. An analysis as in Case 7 gives as possibilities:

$$\lambda = \tfrac{1}{2}, \quad \text{no extra conditions on } y, z, t$$

$$\lambda \neq \tfrac{1}{2}, \quad y, z, t \text{ have to be zero.}$$

Case 9. We may assume $(\eta) = (x:x: -x: -x:z: -z)$. If $x = 0$ we are back in Case 5, so we may take $x = 1$. The equations $P_\lambda(1) = P_\lambda(-1) = P_\lambda(z) = P_\lambda(-z) = 0$ reduce to:

$$1 - z^4 = 2\lambda(2 - z^2 - z^4)$$

$$1 + z^2 = 2\lambda(z^2 + 2).$$

But the first equation follows from the second, and for each value of λ we find two values for z as solution of

$$(2\lambda - 1)z^2 + (4\lambda - 1) = 0.$$

We can summarize the results of the above analysis in the following theorem:

THEOREM 2. *Consider the pencil of* Σ_6-*invariant varieties* $\mathcal{M}_{(\alpha:\beta)} \in P^4$, *defined by*

$$\mathcal{M}_{(\alpha:\beta)} : \alpha S_5 + \beta S_2 S_3 = 0$$

Then: A. For a general value of $(\alpha:\beta)$ $\mathcal{M}_{(\alpha:\beta)}$ has exactly 100 singular points. These points are the Σ_6-orbits of:

$$(1:1:1:-1:-1:-1), \quad \text{the 10 } Segre \text{ nodes}$$

$$(1:1:-1:-1:z:-z), \quad \text{the 90 } Moving \text{ nodes}$$

where z is a solution of $\beta z^2 + (\alpha + 2\beta) = 0$.

B. For 6 points $(\alpha:\beta) \in P^1$ the singular locus $\Sigma_{(\alpha:\beta)}$ of $\mathcal{M}_{(\alpha:\beta)}$ is different. The following table summarizes the situation:

$(\alpha:\beta)$	$\Sigma_{(\alpha:\beta)}$	Point of Σ_6-orbit
	106 nodes (10)	$(1:1:1:-1:-1:-1)$
$(25:1)$	(90)	$(1:1:-1:-1:3\sqrt{-3}:-3\sqrt{-3})$
	(6)	$(1:1:1:1:1:-5)$
	130 nodes (10)	$(1:1:1:-1:-1:-1)$
$(1:1)$	(90)	$(1:1:-1:-1:\sqrt{-3}:-\sqrt{-3})$
	(30)	$(1:1:1:1:\sqrt{-3}-2:-\sqrt{-3}-2)$
$(-3:1)$	10 Del Pezzo nodes	$(1:1:1:-1:-1:-1)$
$(0:1)$	the surface $S_2 = S_3 = 0$	
$(-2:1)$	10 nodes and 15 lines	$(1:1:1:-1:-1:-1)$ $(x:x:y:y:z:z)$ $(x+y+z=0)$
$(1:0)$	10 nodes and 20 lines	$(1:1:1:-1:-1:-1)$ $(0:0:0:y:z:t)$ $(y+z+t=0)$

Remarks. The 10 Segre points are also the singular point of the Segre Cubic. On each of the 45 lines connecting two Segre nodes we find a pair of points, moving over these lines as we vary $(\alpha:\beta)$, whence the name *moving nodes*. The above computation does not show that these point are really nodes. This has to be checked in each case. However, for the variety $\mathcal{M}$ there is an argument that all 130 singularities are nodes, because if one point were not a node, then by symmetry we would have a Σ_6-orbit and this is not allowed by the spectral bound. To be more precise, the spectrum of the cone over a smooth quintic in P^3 is as follows:

Spectral number	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	$\frac{8}{5}$	$\frac{9}{5}$	$\frac{10}{5}$	$\frac{11}{5}$	$\frac{12}{5}$	$\frac{13}{5}$	$\frac{14}{5}$	$\frac{15}{5}$	$\frac{16}{5}$
Multiplicity	1	4	10	20	31	40	44	40	31	20	10	4	1

The spectral bound is that the total number of spectral numbers of all the singularities of a quintic $\subset P^4$ in *any* interval $(a, a+1)$ is less than the number of spectral numbers of the above spectrum in the same interval. If we take the interval $(\frac{6}{5}, \frac{11}{5})$, we find $N_4(5) \leq 135$. Now any singularity worse than A_1 is adjacent to A_2, which implies that in any open interval of length one that contains $[\frac{11}{6}, \frac{13}{6}]$, there are at least two spectral numbers of this singularity. But $(\frac{6}{5}, \frac{11}{5})$ is such an interval. If, say, the smallest Σ_6-orbit would consist of

singularities worse than a node, then we would find at least 140 spectral numbers in this interval, contradicting the spectral bound. Hence, all singularities have to be nodes.

It can be checked that the singularity transverse to a general point of a singular line of $\mathcal{M}_{(-2:1)}$ and of $\mathcal{M}_{(1:0)}$ is also an ordinary double point.

The variety $\mathcal{M}_{(-3:1)}$ has only 10 singular points, but these are not ordinary double points. For this value of $(\alpha:\beta)$ the 90 moving nodes coalesce in 9-tuples with the 10 Segre nodes. The resulting singularity I call a Del Pezzo node, as it is locally isomorphic to the cone over the cubic surface in P^3.

The variety $\mathcal{M}_{(1:0)}$ is defined simply by $S_5 = 0$. It is the *Hessian* of the Segre Cubic. In [3] this variety is called *Nieto's threefold* and is related to the moduli space of abelian surfaces with $(1:3)$-polarization.

A similar analysis of the pencil

$$\mathcal{Q}_{(\alpha,\beta)}: \alpha S_4 + \beta S_2^2 = \frac{-\alpha}{4} C_4 + \frac{\alpha + 2\beta}{8} C_2^2 = 0$$

of Σ_6-invariant quartics in P^4 can be found in [6]. One finds for general $(\alpha:\beta)$ 30 nodes. For $(\alpha:\beta) = (1:0)$ we obtain the Burkhardt quartic with 15 extra nodes, and for $(\alpha:\beta) = (4:-1)$ the projective dual of the Segre Cubic (*Igusa quartic*). It has 15 singular lines. The values $(\alpha:\beta) = (5:1)$ and $(\alpha:\beta) = (3:-1)$ give quartics with resp. 36 and 40 nodes.

§3. ANALOGY OF $\mathcal{M}$ WITH SEGRE CUBIC AND BURKHARDT QUARTIC

The Segre Cubic $\mathcal{S}$ and the Burkhardt Quartic $\mathcal{B}$ have the following remarkable property:

The intersection of the tangent cone at a node with the variety consists of 6 (for $\mathcal{S}$) respectively 8 (for $\mathcal{B}$) planes.

So in the projectivized tangent space P^3 of the ambient P^4 at the node, these planes give lines intersecting as follows:
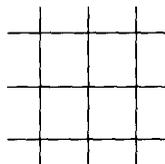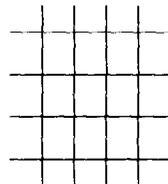
for $\mathcal{S}$ for $\mathcal{B}$

The quintic $\mathcal{M}$ has a similar property:

THEOREM 3. *The intersection of the tangent cone at one of the* 10 *Segre nodes with the variety $\mathcal{M}$ consists of* 10 *planes. In the projectivized tangent space P^3 at such a node the planes give lines intersecting as follows:*
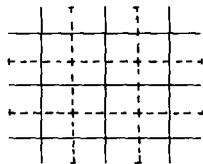
Here the 2 drawn line tripels correspond to 6 of the 15 planes of the Segre Cubic, passing through a node, whereas the dashed lines correspond to 4 of the 40 extra planes on $\mathcal{M}$, passing in 4 tuples through the Segre nodes. In total we have the following planes on $\mathcal{M}$:

A. The 15 *Segre planes*, given by (the Σ_6-orbit of)

$$X_0 + X_3 = 0$$
$$X_1 + X_4 = 0$$
$$X_2 + X_5 = 0.$$

B. The 40 *Extra planes*, given by (the Σ_6-orbit of) the equations:

$$X_0 + X_1 + X_2 + X_3 + X_4 + X_5 = 0$$
$$X_0 + \omega X_1 + \omega^2 X_2 = 0$$
$$X_3 + \omega X_4 + \omega^2 X_5 = 0$$

where $\omega = \dfrac{-1 + \sqrt{-3}}{2}$.

Proof. It is easy to see that the above Segre planes in fact lie on the Segre Cubic $S_3 = 0$ and on $S_5 = 0$. For reasons of degree, the union of the 15 Segre planes is the complete intersection of these two hypersurfaces. It follows that these planes lie on each $\mathcal{M}_{(\alpha:\beta)}$. However, the variety $\mathcal{M}$ contains more planes. First note that the following 16 nodes are in the above extra plane:

0	$(1:1:1:-1:-1:-1)$	7	$(c:-1:1:-c:1:-1)$
		8	$(c:-1:1:-1:-c:1)$
1	$(1:1:1:1:a:b)$	9	$(c:-1:1:1:-1:-c)$
2	$(1:1:1:b:1:a)$	10	$(1:c:-1:-1:-c:1)$
3	$(1:1:1:a:b:1)$	11	$(1:c:-1:1:-1:-c)$
		12	$(1:c:-1:-c:1:-1)$
4	$(a:b:1:1:1:1)$	13	$(-1:1:c:1:-1:-c)$
5	$(1:a:b:1:1:1)$	14	$(-1:1:c:-c:1:-1)$
6	$(b:1:a:1:1:1)$	15	$(-1:1:c:-1:-c:1)$

where $a = 2\omega - 1 = \sqrt{-3} - 2$, $b = 2\omega^2 - 1 = -\sqrt{-3} - 2$ and $c = 2\omega + 1 = \sqrt{-3}$. The point 0 is the unique Segre node in this plane. One sees that the following four-tuples of points are on a line: (0; 1, 2, 3), (0; 4, 5, 6), (0; 7, 10, 13), (0; 8, 11, 14), (0; 9, 12, 15). Apart from these 5 lines through the Segre node, there are 9 lines of the type (1, 4; 9, 10), etc., containing two extra and two moving nodes. As each of these 14 lines contain 4 nodes, they have to be contained in $\mathcal{M}$. As we have more than 5 of such lines all lying in the plane, we conclude that the whole plane has to be contained in $\mathcal{M}$. The configuration of the 10 planes intersecting in a Segre node obviously lies on a quadric cone, which has to be the tangent cone of $\mathcal{M}$ at the node. We can conclude that for reasons of degree these 10 planes together form the complete intersection of the quadric cone and $\mathcal{M}$.

CONCLUDING REMARKS

The variety $\mathcal{M}$ does not have this above property with respect to the other 120 nodes. This is in contrast with the Segre Cubic, and the Burkhardt Quartic. In the Burkhard case, there is an extra symmetry, relating the nodes of the two Σ_6-orbits, making the variety invariant under the simple group of order 25920. No such thing can happen for $\mathcal{M}$.

One can prove that the *defect* of $\mathcal{M}$ is 29. This implies that for a small resolution $\tilde{\mathcal{M}}$ of $\mathcal{M}$ one has: $\dim(H^2(\tilde{\mathcal{M}})) = 30$, $\dim(H^3(\tilde{\mathcal{M}})) = 2$.

(Note that $H^3(\tilde{\mathcal{M}})$ is also equal to the weight-three part $Gr_3^W(H^3(\mathcal{M}))$ of the mixed Hodge structure $H^3(\mathcal{M})$.) In particular, we see that $\tilde{\mathcal{M}}$ is *rigid*. The same holds for the 126 nodal Hirzebruch quintic and the 125 nodal Schoen quintic (see [10]). The L-function of $H^3(\tilde{\mathcal{M}})$ is equal to the L-function of the unique *weight four cusp form for the group* $\Gamma_0(6)$. This can be checked by counting points modulo p and comparing with a table of Fourier coefficients of modular forms. (I thank B. van Geemen for doing this calculation.) The same L-function is associated to the varieties $\mathcal{M}_{(-2:1)}$ and $\mathcal{M}_{(1:0)}$. These facts suggest that there are correspondences of these varieties with the elliptic modular threefold of $\Gamma_0(6)$. However, $\mathcal{M}_{(-3:1)}$ gives rise to a weight four form for $\Gamma_0(21)$. Also, it seems to be of interest to study the Picard-Fuchs equation for this family, as there seems to be a rank four piece of the cohomology splitting off. These matters will be discussed in a subsequent paper.

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